ON THE POINTS OF WEIERSTRASS IN
DIMENSIONS GREATER THAN ONE[1]

BY

ROY H. OGAWA

ABSTRACT. In this paper, the classical concept of Weierstrass points on a Riemann surface is generalized to the consideration of similar points associated with a holomorphic vector bundle E over a compact complex manifold M. These points are invariants of the pair (E, M). The study of these general Weierstrass points is then initiated in this paper by deriving some results about the relationship of the points to singular sets of holomorphic mappings of the manifold to Grassmann spaces associated with the vector space of sections of the vector bundle. The accessibility of the point sets are demonstrated with some examples.

0. **Introduction.** The purpose of this paper is to generalize somewhat the definition of Weierstrass points used in the theory of compact Riemann surfaces, and the work of J. Lewittes (see [Lw]) in showing them to be related to a sequence of Kähler metrics which he constructed on Riemann surfaces.

The classical theory of Weierstrass points has played an interesting role in the study of Riemann surfaces. Especially when related to the principal mapping of Noether.

If M is a compact Riemann surface of genus $g > 1$, its canonical bundle K has a g-dimensional space of sections $\Gamma(K)$. For any p in M, we can choose a basis $\omega_1, \cdots, \omega_g$ which has orders of vanishing at p

$$(1) \qquad 0 = \nu_p(\omega_1) < \nu_p(\omega_2) < \cdots < \nu_p(\omega_g) < 2g - 1.$$

p is a Weierstrass point if $\nu_p(\omega_g) > g - 1$. (See Gunning [G] for further details.)

These points are finite in number, the number being between $2(g + 1)$ and $(g - 1)g(g + 1)$.

They are invariant under holomorphic automorphisms of M.

For each $p \in M$, there is always a section in $\Gamma(K)$ which does not vanish at p, and the subspace of $\Gamma(K)$ consisting of sections which vanish at p has dimension $g - 1$.

Consider the map of M into the Grassmannian of $g - 1$ dimensional

subspaces of $\Gamma(K)$, which is the projective space $P(\Gamma(K))$, defined by taking each p in M to the subspace of $\Gamma(K)$ which vanishes at p. This map is holomorphic and is called the principal mapping of M. The singularities of this mapping are the points where, in the sequence (1), $\nu_p(\omega_2) > 1$. These points are certainly Weierstrass points.

The more recent work of Lewittes gave geometric meaning to the other Weierstrass points. He constructed a sequence of generalizations of the Bergman-Kobayashi metrics on Riemann surfaces. The jth (where $1 \le j \le g - 1$) is defined at points p where, in the sequence (1), $\nu_p(\omega_j) = j - 1$.

The points where the curvature of the metric is negative are those p where in addition, in the sequence (1), $\nu_p(\omega_{j+1}) = j$. All other points have zero curvature.

In this paper, we generalize the definition of Weierstrass points to the situation of a compact complex manifold of dimension n, and an arbitrary vector bundle E, as long as $\Gamma(E)$ is nontrivial. This is done in §2, after a review of the definitions and properties of jet bundles in §1.

In §3, we consider associated principal mappings into Grassmannians where we prove our main theorem relating the singular sets of the mappings to the sets of Weierstrass points. We point out the relationship with the work of Lewittes, a little of which was mentioned above. Actually the metrics we discuss are his curvatures. His situation on a surface lends itself to more structure for the money than we can buy in higher dimension.

In §4, we give some examples, showing the accessibility of the points to computation.

The author wishes to acknowledge his debt to his friend and advisor, Professor S. Kobayashi.

1. **The k-jet bundle $J^k(E)$.** In this section we briefly review the construction that is basic to the discussions of this paper. For more details and proofs, see Chapter IV of Palais [Pa].

1.1. We use the multi-index notation throughout:

$\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ where each μ_i is a nonnegative integer

$z^\mu = z_1^{\mu_1} z_2^{\mu_2} \cdots z_n^{\mu_n}$

$dz^\mu = (dz_1)^{\mu_1}(dz_2)^{\mu_2} \cdots (dz_n)^{\mu_n}$ (symmetric product)

$|\mu| = \Sigma_{i=1}^n \mu_i$

$\mu! = \Pi_{i=1}^n \mu_i!$

$C(\mu, \sigma) = (\mu - \sigma)! / \mu! \sigma!$

1.2. Let M be a complex manifold of dimension n and Ω be the sheaf of germs of holomorphic functions on M with stalk Ω_p over p in M. Let $\pi: E \to M$

be a holomorphic vector bundle over M with fiber rank r and $\Omega(E)$ be the sheaf of germs of holomorphic sections of E over M with stalk $\Omega_p(E)$ over p in M. $\Omega(E)$ is a locally free analytic sheaf of Ω-modules over M.

Fix an arbitrary point p in M. Let m_p be the unique maximal ideal of Ω_p consisting of all the germs which vanish at p, i.e. the nonunits of the integral domain Ω_p. If U is a coordinate neighborhood of p with coordinate function $z = (z_1, z_2, \cdots, z_n)$ centered at p, i.e. $z(p) = (0, 0, \cdots, 0)$, then the elements of Ω_p are representable by power series $\sum_{|\mu| \geq 0} a_\mu z^\mu$ which converge in some neighborhood of p in U. The elements of m_p are those germs representable by those series of the form $\sum_{|\mu| \geq 1} a_\mu z^\mu$.

1.3 For integers $k \geq -1$, we consider the ideal m_p^{k+1}, where $m_p^0 = \Omega_p$. The elements of m_p^{k+1} are those in Ω_p represented by $\sum_{|\mu| \geq k} a_\mu z^\mu$.

Define the submodule Z_p^k of $\Omega_p(E)$ by $Z_p^k = m_p^{k+1} \cdot \Omega_p(E)$.

Now define $J_p^k(E) = \Omega_p(E)/Z_p^k$ for $k \geq -1$. We are identifying germs of sections of E at p, which, in some local representation, have the same derivatives of order less than or equal to k. These derivatives then give a local trivialization of $J^k(E) = \bigcup_{p \in M} J_p^k(E)$ which makes it a holomorphic vector bundle of fiber rank $R_k = r \cdot C(n + k, n)$ where $C(n + k, n) = (n + k)!/n!k!$. Note that $J^{-1}(E) = 0$ and that $J^0(E) = E$.

1.4 We have the natural homomorphisms

$$j_p^k : \Omega_p(E) \longrightarrow J_p^k(E), \quad j_p^k : \Omega(E) \longrightarrow J_p^k(E), \quad \text{and} \quad j^k : \Gamma(E) \longrightarrow \Gamma(J^k(E))$$

where $\Gamma(E)$ denotes the vector space of holomorphic sections of E over M.

Since $m_p^k \supset m_p^{k+1}$, we have, for each $k \geq -1$, an induced homomorphism of vector bundles

$$\pi^{k+1, k} : J^{k+1}(E) \longrightarrow J^k(E),$$

which is analytic and surjective.

1.5. Letting T^* denote the holomorphic cotangent bundle of M and $S^k(T^*)$ its kth symmetric power, we have an injection (see Palais [Pa])

$$0 \longrightarrow S^{k+1}(T^*) \otimes E \longrightarrow J^{k+1}(E).$$

The homomorphism is defined locally as follows: Let p be a point in M, $z = (z_1, z_2, \cdots, z_n)$ be a local coordinate system centered at p, and $\{v_1, v_2, \cdots, v_r\}$ be a local basis for E near p. Thus each v_i is a section of E in a neighborhood of p in M and the set $\{v_1(q), v_2(q), \cdots, v_r(q)\}$ is a basis for the vector space E_q for q in some neighborhood of p. Map the element $(dz^\mu)_p \otimes v_i(p)$, where $|\mu| = k + 1$, of $(S^{k+1}(T^*) \otimes E)_p$ to the equivalence class in $\Omega_p(E)/Z_p^{k+1}$ represented by the local section $z^\mu v_i$ of E in some neighborhood of p. Extend this map by linearity to get the desired homomorphism of vector bundles.

Clearly (from the construction), the image of $(S^{k+1}(T^*) \otimes E)_p$ is Z_p^k/Z_p^{k+1}, so that we have the short exact sequence

(e1) $$0 \to S^{k+1}(T^*) \otimes E \to J^{k+1}(E) \to J^k(E) \to 0.$$

The discussion also gives the exact sequence

(e2) $$0 \to Z_p^{k+1} \to Z_p^k \to (S^k(T^*) \otimes E)_p \to 0.$$

2. Weierstrass points. Let M be a compact complex manifold of dimension n, E a holomorphic vector bundle on M with fiber rank r, and $\Gamma(E)$ the vector space of holomorphic sections of E. Since M is compact, $\Gamma(E)$ has finite dimension. *We assume that the dimension of $\Gamma(E)$ is nonzero.* Let H be a nontrivial linear subspace of $\Gamma(E)$ of dimension h.

2.1. For each p in M and each integer $k \geq -1$, define H_p^{k+1} to be the kernel of the kth jet evaluation map at p, $j_p^k : H \to J_p^k(E)$. We remark that this is the inverse image of Z_p^k in the homomorphism $H \to \Omega_p(E)$ which takes the section to its germ at p. The definition is equivalent to the exactness of

(e3) $$0 \to H_p^{k+1} \to H \to J_p^k(E)$$

which parallels (e1).

Proposition. (i) *For $k \geq -1$, $H_p^{k+1} \supset H_p^{k+2}$.*
(ii) *There is an exact sequence*

(e4) $$0 \to H_p^{k+2} \to H_p^{k+1} \to (S^{k+1}(T^*) \otimes E)_p.$$

Proof. (1) is clear from the remark after the definition and the fact that $Z_p^k \supset Z_p^{k+1}$.

First we define the last homomorphism in (e4) to be the dashed arrow defined by the diagram:

$$
\begin{array}{ccccccccc}
0 & \to & H^{k+2} & \to & H_p^{k+1} & & & & \\
 & & \downarrow & & \downarrow & \searrow & & & \\
0 & \to & Z^{k+1} & \to & Z^{k+2} & \to & (S^{k+1}(T^*) \otimes E)_p & \to & 0
\end{array}
$$

The exactness of the sequence (e4) then follows by examining the above diagram and using the remark after the definition of H_p^{k+1}.   Q.E.D.

The exact sequence (e4) parallels (e2).

2.2. For those k, satisfying $R_k \leq h$ (where $R_k = rC(n + k, n)$), define

$$W_k(H) = \{p \in M \mid j_p^k : H \to J_p^k(E) \text{ is not surjective}\}.$$

So for p *not* in $W_k(H)$ we have the exactness of

(e5) $$0 \to H_p^{k+1} \to H \to J_p^k(E) \to 0.$$

We also have the

Proposition. *For p not in $W_k(H)$, where $k \geq 0$, we have the exact sequence*

(e6) $$0 \to H_p^{k+1} \to H_p^k \to (S^k(T^*) \otimes E)_p \to 0.$$

Proof. We start with (e4) and show surjectivity of the last homomorphism. Consider the diagram

$$
\begin{array}{c}
0 \\
\downarrow \\
(S^k(T^*) \otimes E)_p \\
\downarrow \\
0 \to H_p^{k+1} \to H \xrightarrow{j_p^k} J_p^k(E) \to 0 \\
\downarrow \\
J_p^{k-1}(E) \\
\downarrow \\
0
\end{array}
$$

The image of H_p^k by j_p^k in $J_p^k(E)$ is identical with the image of $(S^k(T^*) \otimes E)_p$ by the injection.   Q.E.D.

For those k, satisfying $R_k > b$, define

$$W_k(H) = \{ p \in M \mid j_p^k : H \to J_p^k(E) \text{ is not injective} \}.$$

Generally then we will say that $W_k(H)$ is the set of p in M where $j_p^k : H \to J_p^k(E)$ is not of maximal rank. $W_k(H)$ is called the kth order Weierstrass set for M, E, and H.

2.3. Corollary. (i) $W_{-1}(H) = \emptyset$.
  (ii) *If $R_k < R_{k+1} \leq b$, then $W_k(H) \subset W_{k+1}(H)$.*
  (iii) *If $b < R_k < R_{k+1}$, then $W_k(H) \supset W_{k+1}(H)$.*
  (iv) *If $H_1 \subset H_2$ and $R_k \leq b_1 < b_2$, then $W_k(H_1) \supset W_k(H_2)$.*
  (v) *If $H_1 \subset H_2$ and $b_1 < b_2 < R_k$, then $W_k(H_1) \subset W_k(H_2)$.*

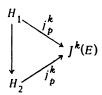**Proof.** Since $J_p^{-1}(E) = 0$, (i) is obvious.
The diagram

$$
\begin{array}{c}
H \to J_p^{k+1}(E) \\
\searrow \quad \downarrow \\
J_p^k(E)
\end{array}
$$

gives (ii) and (iii) immediately.

The diagram

$$
\begin{array}{ccc}
H_1 & \xrightarrow{\ j_p^k\ } & \\
\downarrow & \searrow & J^k(E) \\
H_2 & \nearrow{\ j_p^k\ } &
\end{array}
$$

gives (iv) and (v) immediately.   Q.E.D.

Thus if k' is the greatest integer satisfying $R_{k'} \le b$, we have the two chains

$$\varnothing = W_{-1}(H) \subset W_0(H) \subset \cdots \subset W_{k'}(H)$$

and

$$W_{k'+1}(H) \supset W_{k'+2}(H) \supset \cdots .$$

It is quite clear that there need be no inclusion relationship between $W_{k'}(H)$ and $W_{k'+1}(H)$.

2.4. **Proposition.** *The $W_k(H)$ are analytic subvarieties of M.*

Proof. In fact, consider the trivial bundle $M \times H$ and the obvious bundle map $j^k : M \times H \to J^K(E)$. Then for $m = \min(R_k, b)$, take the mth exterior power bundle map $\bigwedge^m j^k : \bigwedge^m (M \times H) \to \bigwedge^m J^k(E)$. This gives a section $\bigwedge^m j^k$ of the bundle

$$\operatorname{Hom}(\textstyle\bigwedge^m (M \times H),\ \bigwedge^m J^k(E)) \simeq (\bigwedge^m (M \times H))^* \otimes \bigwedge^m J^k(E)$$

whose zero set is $W_k(H)$. (The asterisk denotes the dual bundle.)   Q.E.D.

2.5. We remark that the sets $W_k(H)$ have a finer structure. For each k and integer u satisfying $0 \le u \le R_k$, define

$$W_k^u(H) = \{p \in M \mid \operatorname{rank}(j_p^k : H \to J_p^k(E)) < \min(u, b)\}.$$

Then $W_k(H) = W_k^u(H)$ for $u = R_k$, and $W_k^u(H) \subset W_k^{u+1}(H)$ for $u < u + 1 \le R_k$.

Proposition. *The $W_k^u(H)$ are analytic subvarieties of M.*

Proof. As above, $W_k^u(H)$ is the zero set of the section

$$\textstyle\bigwedge^m j^k \in \Gamma((\bigwedge^m (M \times H))^* \otimes \bigwedge^m J^k(E))$$

where $m = \min(u, b)$.   Q.E.D.

3. **Maps into Grassmannians, semi-Kähler metrics.** In this section, we again assume that we have a compact complex manifold M, a holomorphic vector bundle E with *nontrivial* vector space of holomorphic sections $\Gamma(E)$, and a *nontrivial* subspace H of $\Gamma(E)$. We then construct a mapping of $M - W_k(H)$ into a Grassmann

space associated with H. By pulling back to $M - W_k(H)$, any metric in the Grass-mann space then gives a semimetric on $M - W_k(H)$.

3.1. First, we need notation for Grassmann spaces. Let V be a vector space of dimension h. Let x, y be nonnegative integers with $x + y = h$. Let $\mathrm{Gr}_{x,y}(V)$ be the Grassmann manifold of x dimensional subspaces of V.

We construct the standard atlas on $\mathrm{Gr}_{x,y}(V)$. Let V', and V'' be complemen-tary subspaces of V of dimensions x, and y respectively, so that $V = V' \oplus V''$. Consider the set $U_{V',V''}$ of x dimensional subspaces S of V such that the pro-jection of V onto V' maps S bijectively onto V'. Then assign to each such S the element in the space $L(V', V'')$ of linear mappings of V' into V'' which is the composition of the linear mapping $V' \to S$, inverse to the isomorphism given by the projection of V onto V', and the linear mapping $S \to V''$ which is the restric-tion of the projection of V onto V''. This mapping $U_{V'V''} \to L(V', V'') \cong V'^{*} \otimes V''$, together with any isomorphism of $V'^{*} \otimes V''$ with $\mathbf{C}^{xy}$ gives a chart for an atlas for $\mathrm{Gr}_{x,y}(V)$, the other charts coming from varying V' and V'', making it a compact complex manifold of dimension xy.

Let $B_{x,y}(V)$ be the vector bundle of rank x over $\mathrm{Gr}_{x,y}(V)$ whose points con-sist of pairs (S, v), in the trivial bundle $G_{x,y}(V) \times V$, for which v is in S. This is the so called tautological vector bundle over $\mathrm{Gr}_{x,y}(V)$. Letting $Q_{x,y}(V)$ de-note the quotient bundle $\mathrm{Gr}_{x,y}(V) \times V / B_{x,y}(V)$, we get the following bundle exact sequence

$$0 \to B_{x,y}(V) \to \mathrm{Gr}_{x,y}(V) \times V \to Q_{x,y}(V) \to 0$$

over $\mathrm{Gr}_{x,y}(V)$, called the canonical sequence.

3.2. If V is equipped with a hermitian metric, then $\mathrm{Gr}_{x,y}(V)$ is given a Kähler metric defined by the fundamental form ω which is defined as follows. We write the representative of ω in $U_{V',V''}$ by choosing orthonormal bases for V' and V'' and then defining

$$(*) \qquad \omega_{V',V''}(S) = -4\sqrt{-1}\ \partial\bar{\partial}\ \log \det(I_{V'} + {}^{t}\bar{Z}'(S)Z'(S))$$

where $I_{V'}$ is the identity matrix, $Z'(S)$ is the matrix of the map $Z(S): V' \to V''$ and ${}^{t}\bar{Z}'(S)$ is the matrix of the adjoint of $Z(S)$, the matrices being representatives with respect to the chosen bases. This C^{∞} form ω of type (1.1) is independent of choice of orthonormal basis, is positive definite, and defines a Kähler metric on $\mathrm{Gr}_{x,y}(V)$.

3.3. Return now to the consideration of an n-dimensional compact complex manifold M, a vector bundle E of rank r over M, the vector space $\Gamma(E)$ of holo-morphic sections of E over M, which we *assume to be nontrivial*, and a subspace H of $\Gamma(E)$ of dimension $h \neq 0$.

For each k with $R_k < h$ and p in $M - W_k(H)$ we have the $h - R_k$ dimensional

subspace H_p^{k+1} of H. This defines on the one hand, a holomorphic vector bundle H^{k+1} of rank $b - R_k$ over $M - W_k(H)$, whose fiber over p in $M - W_k(H)$ is H_p^{k+1}, and on the other hand a holomorphic mapping

$$N_k(H): M - W_k(H) \longrightarrow \mathrm{Gr}_{b - R_k, R_k}(H)$$

called the kth associated principal mapping. We have the following commutative diagram:

$$
\begin{array}{ccc}
H^{k+1} = N_k(H)^{-1} B_{b - R_k, R_k}(H) & \longrightarrow & B_{b - R_k, R_k}(H) \\
\downarrow & & \downarrow \\
M - W_k(H) & \longrightarrow & \mathrm{Gr}_{b - R_k, R_k}(H)
\end{array}
$$

3.4. At the points in $M - W_k(H)$ where $N_k(H)$ is an immersion, any metric on the image will be pulled back to a metric.

Theorem 3.1. *For $k \geq 0$, with $R_{k+1} \leq b$, let $S_k(H)$ be the set of singularities of the map $N_k(H)$. Then we have the inclusion $S_k(H) \subset W_{k+1}(H)$.*

Proof. Let p be a point of $M - W_{k+1}(H)$. We examine $dN_k(H)(p)$.

Choose a coordinate system in a neighborhood G of p, centered at p. Thus $z = (z_1, z_2, \cdots, z_n)$ is defined in G and $z(p) = (0, 0, \cdots, 0)$.

Since p is in $M - W_{k+1}(H)$, it is in $M - W_m(H)$ for all $0 \leq m \leq k + 1$, in particular, for $m = 0$. Thus there are sections $\Phi_1, \cdots, \Phi_r$ or E such that the values $\Phi_1(p), \cdots, \Phi_r(p)$ are linearly independent in E_p and gives a basis of E_p. This is an open condition so that $\Phi_1(q), \cdots, \Phi_r(q)$ gives a basis of E_q for all q in some neighborhood of p. Let G be made smaller so that this is true in G.

Choose a basis for H in the following manner. First include the elements $\Phi_1, \cdots, \Phi_r$. Again, all of the $j_p^m : H \to J_p^m(E)$ are surjective for m, $0 \leq m \leq k + 1$, since p is not in $W_{k+1}(H) \supset W_k(H) \supset \cdots \supset W_0(H)$. For all such m then, we have (e6)

$$0 \longrightarrow H_p^{m+1} \longrightarrow H_p^m \longrightarrow (S^m(T^*) \otimes E)_p \longrightarrow 0.$$

For each such m then, we can choose $\Phi'_{R_{m-1}+1}, \Phi'_{R_{m-1}+2}, \cdots, \Phi'_{R_m}$ to be elements of H_p^m whose images in $(S^m(T^*) \otimes E)_p$ are the elements $(dz^\mu)_p \otimes \Phi_i(p)$ for all μ with $|\mu| = m$ and $1 \leq i \leq r$, ordered lexicographically in the $(n + 2)$-tuple $(i, |\mu|, \mu)$.

Then define $\Phi_{R_k+1}, \Phi_{R_k+2}, \cdots, \Phi_{R_{k+1}}$ to be

$$\Phi'_{R_k+1}, \Phi'_{R_k+2}, \cdots, \Phi'_{R_{k+1}}.$$

Next, starting with $m = k$ and decreasing m until $m = 1$, define for u, $R_{m-1} + 1 \leq u \leq R_m$,

$$\Phi_u = \Phi'_u - \sum a_{uv}\Phi_v,$$

where the summation is over all v, $R_m \leq v \leq R_{k+1}$ and the a_{uv} are the constants chosen so that if u corresponds to the $(n+2)$-tuple (i, m, μ) then $\Phi_u - z^\mu\Phi_i$ is in H_p^{k+2}. (In fact if v corresponds to the $(n+2)$-tuple $(j, |\nu|, \nu)$, then a_{uv} is the coefficient of the $z^\nu\Phi_j$ term of the expansion of Φ'_u in z, and the $\Phi_1, \cdots, \Phi_r$.)

Finally, choose $\Phi_{R_{k+1}+1}, \Phi_{R_{k+1}+2}, \cdots, \Phi_h$ to be a basis for H_p^{k+2}.

Let $H' = H_p^{k+1}$ and H'' be the subspace of H spanned by $\Phi_1, \cdots, \Phi_{R_k}$. Thus $j_p^k|_{H''}$ is an isomorphism of H'' with $J_p^k(E)$.

Now, if p' is in G we need to know the relationship between $H_{p'}^{k+1}$ and H_p^{k+1} in order to construct the local coordinates of the $H_{p'}^{k+1}$ in the standard chart in $U_{H', H''}$. Let G be made small enough so that $H_{p'}^{k+1}$ is in $U_{H', H''}$ for all p' in G. Then, to construct the linear map of H' into H'' corresponding to $H_{p'}^{k+1}$ for each p' in G, we will take a basis of $H_{p'}^{k+1}$, split its elements into H' and H'' components, and then map each H' component into the corresponding H'' component. In more detail, for all v, $R_k \leq v \leq h$,

$$\widetilde{\Phi}_v = \Phi_v + \sum_{u=1}^{R_k} Z_{uv}(p')\Phi_u$$

where the $Z_{uv}(p')$ are constants chosen so that the $\widetilde{\Phi}_v$ lie in $H_{p'}^{k+1}$ and thus will span $H_{p'}^{k+1}$. That this is possible is a restatement of the fact that $H_{p'}^{k+1}$ and H are complementary direct summands of H, that the Φ_v have the $\widetilde{\Phi}_v$ as projections in $H_{p'}^{k+1}$, and that the $\widetilde{\Phi}_v$ span $H_{p'}^{k+1}$ since the Φ_u, $1 \leq u \leq R_k$, span H''. Then the map $H' \rightarrow H''$ in $L(H', H'')$ corresponding to $H_{p'}^{k+1}$ is defined by linearly extending the map of the basis elements

$$\Phi_v \rightarrow \sum_{u=1}^{R_k} Z_{uv}(p')\Phi_u, \qquad R_k \leq v \leq h.$$

This has the matrix $(Z_{uv}(p'))$.

Next, we need to calculate, in the local coordinates $z = (z_1, z_2, \cdots, z_n)$ centered at p, the $n \times R_k(h - R_k)$ Jacobian matrix

$$(D^j Z_{uv}(p)), \qquad 1 \leq j \leq n, \ 1 \leq u \leq R_k, \ R_k < v \leq h.$$

We accomplish this by the technique of implicit differentiation.

First, let

$$\Phi_v(q) = \sum_{i=1} \sum_{|\mu| \geq 0} a^i_{v\mu} z(q)^\mu \Phi_i(q), \qquad 1 \leq v \leq h,$$

where the convergences are good in (a possibly smaller) G.

Then

$$\Phi_v(q) = \sum_{i=1}^{r} \sum_{|\mu| \geq 0} \alpha^i_{v\mu}(q)^\mu \Phi_i(q) + \sum_{u=1}^{R_k} Z_{uv}(p) \sum_{i=1}^{r} \sum_{|\mu| \geq 0} \alpha_{u\mu} z(q)^\mu \Phi_i(q)$$

for $R_k < v \leq b$. But $\Phi_v(q)$ vanishing of order k at p' implies that, for each multi-index σ, with $|\sigma| \leq k$, each i, and each v, $R_k < v \leq b$,

$$0 = D^\sigma \left\{ \sum_{|\mu| \geq 0} \alpha^i_{v\mu} z(q)^\mu + \sum_{u=1} Z_{uv}(p') \sum_{|\mu| \geq 0} \alpha^i_{u\mu} z(q)^\mu \right\}(p')$$

$$= \sum_{\mu \geq \sigma} \alpha^i_{v\mu} C(\mu, \sigma) z(p')^{\mu - \sigma} + \sum_{u=1} Z_{uv}(p') \sum_{\mu \geq \sigma} \alpha_{u\mu} C(\mu, \sigma) z(p')^{\mu - \sigma}.$$

Now, differentiate with respect to $z_j(p')$ for each j, $1 \leq j \leq n$, and evaluate at p to get

$$0 = \sum_{\mu \geq \sigma} \alpha^i_{v\mu}(\mu_j - \sigma_j) C(\mu, \sigma) z(p)^{\mu - \sigma - \delta_j} + \sum_{u=1} D^j Z_{uv}(p) \sum_{\mu \geq \sigma} \alpha^i_{u\mu} C(\mu, \sigma) z(p)^{\mu - \sigma}$$

$$+ \sum_{u=1}^{R_k} Z_{uv}(p) \sum_{\mu \geq \sigma} \alpha^i_{u\mu}(\mu_j - \sigma_j) C(\mu, \sigma) z(p)^{\mu - \sigma - \delta_j}.$$

But, since $Z_{uv}(p) = 0$ for all u, v and $z_j(p) = 0$ for all j, $1 \leq j \leq n$, we have the equations

$$0 = \alpha^i_{v\sigma + \delta_j} C(\sigma + \delta_j, \sigma) + \sum_{u=1}^{R_k} D^j Z_{uv}(p) \alpha^i_{u\sigma}$$

or

$$0 = (\sigma_j + 1) \alpha^i_{v\sigma + \delta_j} + \sum_{u=1}^{R_k} D^j Z_{uv}(p) \alpha^i_{u\sigma}.$$

satisfied by the components $D^j Z_{uv}(p)$ of the Jacobian matrix.

We are interested then in the rank of this Jacobian matrix.

First, by our choice of basis, if u corresponds to the $(n + 2)$-tuple $(l, |\lambda|, \lambda)$, then

$$\alpha^i_{u\sigma} = \begin{cases} 1 & \text{if } i = l, \ \lambda = \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

so that we get

$$D^j Z_{uv}(p) = -(\sigma_j + 1) \alpha^i_{v\sigma + \delta_j}$$

where u corresponds to $(i, |\sigma|, \sigma)$. Also, since $\alpha^i_{v\sigma + \delta_j} = 0$ if $|\sigma + \delta_j| < k + 1$, we

have $D^j Z_{uv}(p) = 0$ for u corresponding to $(i, |\sigma|, \sigma)$ and any v, when $|\sigma + \delta_j| < k + 1$, i.e. $|\sigma| < k$.

Finally, when $v > R_{k+1}$ the element Φ_v is in H_p^{k+2}, and we have $\alpha^i_{v\sigma + \delta_j} = 0$ for $|\sigma| \le k$.

So the only possible nonzero columns of the Jacobian matrix $(D^j Z_{uv}(p))$ are those for the pairs u and v with u corresponding to (i, k, σ) and v satisfying $R_k < v \le R_{k+1}$. These number

$$(R_k - R_{k-1})(R_{k+1} - R_k) = r^2 \cdot C(n - 1 + k, n - 1)C(n + k, n - 1)$$

in all.

Clearly, if $R_{k+1} \le h$ and p is not in $W_{k+1}(H)$, there are at least n linearly independent columns, the Jacobian matrix is of rank n, and p is not in $S_k(H)$. Q.E.D.

From the proof of the theorem we can conclude the following.

Corollary 3.2. *If the vector bundle E is in fact a line bundle, and $h \ge R_1$, then $S_0(H) = W_1(H)$.*

Corollary 3.3. *If the manifold is of dimension $n = 1$, and $h \ge R_k + 1 = rk + r + 1$, then $S_k(H) = W^u_{k+1}(H)$ where $u = rk + r + 1$. (See §2 for the definition of $W^u_{k+1}(H)$.)*

Corollary 3.4. *If the manifold is of dimension $n = 1$, the vector bundle E is is a line bundle, and $h \ge R_k + 1 = k + 2$, then $S_k(H) = W_{k+1}(H)$.*

3.5. Now, if H is equipped with a hermitian metric, $\mathrm{Gr}_{h - R_k, R_k}(H)$ can be given the Kähler metric $(*)$, and the pullback ω_k of ω to $M - W_k(H)$ induces on it a Kähler semimetric, positive definite on $(M - W_k(H)) - S_k(H)$.

In particular, if E is the canonical line bundle K, the vector space $\Gamma(K)$ of its holomorphic sections is equipped with a natural hermitian metric given by

$$\langle \sigma_1, \sigma_2 \rangle = \int_M \sigma_1 \wedge \bar{\sigma}_2, \qquad \sigma_1, \sigma_2 \text{ in } \Gamma(k).$$

In the case $k = 0$, the induced semi-Kähler metric on $M - W_0(H)$ is the Kobayashi generalization of the Bergman metric. (See Kobayashi [K].)

If M is a curve, E is a power of the canonical bundle, and H is a Hilbert space, then the Kähler semimetrics defined by ω_k are those defined by Lewittes. (See [Lw].)

If M is arbitrary and E is of the form $E_0 \otimes K$ where E_0 is a holomorphic vector bundle equipped with a hermitian metric, $\Gamma(E)$ can be equipped with a hermitian metric by $\langle \sigma_1, \sigma_2 \rangle = \int_M (\sigma_1, \sigma_2)$, σ_1, σ_2 in $\Gamma(E)$, where (σ_1, σ_2) is the section of $K \otimes \bar{K}$ defined by taking the scalar product in the fibers of E_0 with values in the fibers of $K \otimes \bar{K}$.

4. Examples.

4.1. Now we examine some examples. The first examples deal with line bundles on Riemann surfaces and we need some preparation.

First, we get a result stronger than those of §4, since for Riemann surfaces, the notion of "order of vanishing of a holomorphic function" works well.

So, suppose that M is a compact Riemann surface of genus g and that E is a holomorphic line bundle L on M. Then the rank of $J^k(L)$ is $R_k = 1 \cdot C(1 + k, 1) = k + 1$. Taking the highest case $R_k = b$, i.e. $k = b - 1$, we have the result that $W_{b-1}(L)$ is the zero set of the section $\bigwedge^b j^{b-1}$ of the line bundle $(\bigwedge^b (M \times H))^* \otimes \bigwedge^b J^{b-1}(L)$.

However, in the case of a line bundle, we have the following.

Proposition 4.1 (see [G, p. 103]). *If s is a nontrivial holomorphic section of the line bundle L over a compact Riemann surface M, then for the first Chern number of L, $c_1(L)$,*

$$c_1(L) = \sum_{p \in M} \nu_p(s),$$

where $\nu_p(s)$ is the order of vanishing of s at p.

Thus, we have the following:

Proposition 4.2. *If $\bigwedge^b j^{b-1}$ is nontrivial then the number of Weierstrass points (including multiplicity) is*

$$\sum_{p \in M} \nu_p(\bigwedge^b j^{b-1}) = c_1((\bigwedge^b (M \times H))^* \otimes \bigwedge^b J^{b-1}(L)) = (g-1)(b-1)b + bc_1(L).$$

Proof. The first equality is Proposition 4.1.

For the second we have, using elementary properties of c_1, linear algebra, the fact that $c_1(M \times H) = 0$ since $M \times H$ is trivial, and the exact sequence (e1) from §1,

$$c_1((\bigwedge^b (M \times H))^* \otimes \bigwedge^b J^{b-1}(L)) = c_1((\bigwedge^b (M \times H))^*) + c_1(\bigwedge^b J^{b-1}(L)) = c_1(\bigwedge^b J^{b-1}(L))$$

$$= c_1\left(\bigoplus_{i=0}^{b-1} (S^i(K) \otimes L) \right) \quad \text{(since } T^* \to K = \text{canonical line bundle)}$$

$$= \sum_{i=0}^{b-1} c_1(K^i \otimes L) = (b-1)bc_1(K)/2 + bc_1(L) = (b-1)b(g-1) + bc_1(L) \quad \text{Q.E.D.}$$

Note that this formula depends only on b and not on H, except that $\bigwedge^b j^{b-1}$ must be nonzero somewhere.

In the rest of this discussion of examples, we will take $H = \Gamma(E)$, write $\gamma(E)$ for b, and write $W_k(E)$ for the Weierstrass sets of M, E, and $\Gamma(E)$.

Example 1. M is a compact Riemann surface of genus g and $L = K$ is the canonical bundle of M. Here, the definition agrees with the classical definition of Weierstrass points for k, $0 \leq k \leq g - 1$ (see [Lw] and [G]).

The total number of $(g - 1)$th order Weierstrass points for K, counting order is, by Proposition 4.2,

$$(g - 1)(g - 1)g + 2(g - 1)g = (g - 1)g(g + 1),$$

which agrees with the classical result (see [G, p. 123]).

For future use, we define: An integer $k \geq 1$ is a gap number for a point p in M if and only if p is not in $W_{k-1}(K) - W_{k-2}(K)$ $(W_{-1}(K) = \emptyset)$.

There are g gap numbers $\{\rho_i\}$, $1 \leq i \leq g$, at a point p which satisfy $1 = \rho_1 < \rho_2 < \cdots < \rho_g < 2g$ (see [G, p. 120]). p is a Weierstrass point if and only if the gap sequence $\{\rho_i\}$ is not $1 < 2 < \cdots < g$. We call the numbers $\sigma_i, 1 \leq i \leq g$ which are not gap numbers and which satisfy $1 < \sigma_i \leq 2g$, $\sigma_i \leq \sigma_{i+1}$ the nongap numbers of p.

4.2. Before dealing with the next examples, which are of arbitrary line bundles, we review some results on line bundles. (We refer mostly to Gunning [G] for proofs. See especially §§ 7 and 10.)

Proposition 4.3. *Let L be a line bundle on the Riemann surface M. Let s be a nontrivial section of L. Then L is isomorphic with the line bundle associated to the divisor of s.*

Proof. Let $\{U_\alpha\}$ be a covering of M over which L is locally trivialized an $\{\phi_{\alpha\beta}\}$ be the transition functions for L and $\{U_\alpha\}$. Let $\{s_\alpha\}$ be a local representation for s. Thus $s_\alpha = \phi_{\alpha\beta} s_\beta$ and $\phi_{\alpha\beta} = s_\alpha / s_\beta$. But $\{s_\alpha / s_\beta\}$ is the set of transition functions for the line bundle associated to the divisor of S. Q.E.D.

Using the Riemann-Roch theorem, Gunning has the

Proposition 4.4 [G, pp. 111–112]. (a) *If $c_1(L) < 0$ then $\gamma(L) = 0$.*

(b) *If $c_1(L) = 0$ and $L = M \times \mathbf{C}$ is the trivial line bundle, then $\gamma(L) = 1$.*

(c) *If $c_1(L) = 0$ and $L \neq M \times \mathbf{C}$ then $\gamma(L) = 0$.*

(d) *If $0 < c_1(L) < 2g - 2$ then $c_1(L) - (g - 1) \leq \gamma(L) \leq \min (g, c_1(L) + 1)$. Of course, here, $g \neq 0$.*

(e) *If $c_1(L) = 2g - 2$ and $L = K$, then $\gamma(L) = g$.*

(f) *If $c_1(L) = 2g - 2$ and $L \neq K$, then $\gamma(L) = g - 1$. Of course, here, $g \neq 0$.*

(g) *If $c_1(L) > 2g - 2$ then $\gamma(L) = c_1(L) - (g - 1)$.*

Since we are interested only in those L with $\gamma(L) \neq 0$, we rule out (a) and (c). However, in case (a), since $c_1(L) = -c_1(L^{-1})$, we can still study L by studying L^{-1}. We remark here that in case (b), we always have $W_0(M \times \mathbf{C}) = \emptyset$.

Example 2. If M has genus $g = 0$, i.e. $M = \mathbf{P}_1$, then all line bundles on $\mathbf{P}_1$ are of the form ζ^m, $m \in \mathbf{Z}$, where ζ is the dual of the tautological line bundle on $\mathbf{P}_1$, and $c_1(\zeta) = 1$ [G, pp. 115–116]. So $c_1(\zeta^m) = m$.

By Proposition 4.2, all $W_k(\zeta^m)$ are empty, for $1 \leq k \leq m$.

Example 3. If $g = 1$, then for any line bundle L, with $c_1(L) > 0$, case (g) of the proposition gives $\gamma(L) = c_1(L)$. Write d for this common positive integer. According to Proposition 2.1, the number of zeros of $\bigwedge^d j^{d-1}$ is d^2.

If $d = 1$, L must be a line bundle ζ_p associated with a unique (to L) point divisor p in M. Then $W_0(L) = \{p\}$, and $W_k(L) = \varnothing$ for $k \geq 1$.

In order to discuss the case when $d > 1$, we first consider the mth tensor power line bundle $\zeta_p^m = (\zeta_p)^m$, $m \in \mathbf{Z}$, $m > 1$. Then $c_1(\zeta_p^m) = m$ and $\gamma(\zeta_p^m) = c_1(\zeta_p^m) = m$, by (g) of Proposition 4.4.

Proposition 4.5. p *is not in* $W_k(\zeta_p^m)$ *for all* $0 \leq k < m - 1$. p *is in* $W_{m-1}(\zeta_p^m)$. p *is not in* $W_k(\zeta_p^m)$ *for all* $k \geq m$.

Proof. s_p^m is a section in $\Gamma(\zeta_p^m)$, which vanishes only at p with order m.

Let k be an integer satisfying $k \geq 1$. It is known (see [G, p. 121]) that there is a meromorphic function $f_{p,k}$ on M whose only singularity is a pole at p of order exactly k if and only if k is a nongap number. On the other hand, since $g = 1$, M does not have any Weierstrass points for the canonical bundle since the total number is $(g + 1)g(g - 1)$. So the only gap number for any point is 1. Thus the meromorphic functions $f_{p,k}$ exist for every p and every $k > 1$.

So, for $1 \leq k \leq m$, $f_{p,k}s_p^m$ is a homomorphic section of ζ_p^m. It vanishes at p of order $m - k$, where $0 \leq m - k < m - 1$. Thus $j_p^{m-k'} : \Gamma(\zeta_p^m) \to J_p^{m-k'}(\zeta_p^m)$ is surjective for all $m - k'$, $0 \leq m - k' < m - 1$, and p is not in $W_{m-k'}(\zeta_p^m)$ for all such k'.

Since the set of m sections $\{s_p^m, f_{p,2}s_p^m, \cdots, f_{p,m}s_p^m\}$ has distinct vanishing order at p, they must give a basis for $\Gamma(\zeta_p^m)$. Then it is clear that $j^{m-1} : \Gamma(\zeta_p^m) \to J_p^{m-1}(\zeta_p^m)$ is not surjective and that when $k = m$ and $R_m = m + 1 > \gamma(\zeta_p^m) = m$, we have $j_p^m : \Gamma(\zeta_p^m) \to J_p(\zeta_p^m)$ is injective so that p is not in $W_k(\zeta_p^m)$ for $k \geq m$. Q.E.D.

Now, consider an arbitrary line bundle L with $\gamma(L) = c_1(L) = d > 2$. Let b be any point in M.

Consider the line bundle $L\zeta_b^{1-d}$, which has $c_1(L\zeta_b^{1-d}) = 1$. From the case above when $d = 1$, we see that there is exactly one p in M such that $L\zeta_b^{1-d} = \zeta_p$, from which we get $L = \zeta_p\zeta_b^{d-1}$.

If $p = b$ then $L = \zeta_b^d$ and from Proposition 4.5, b is not in $W_k(L)$ when $k \neq d - 1$ and b is in $W_{d-1}(L)$.

If $p \neq b$ then, letting s_p be a nontrivial section of ζ_p, s_p does not vanish at b (it vanishes exact by at p). Thus, considering the basis s_b^{d-1}, $f_{b,2}s_b^{d-1}, \cdots, f_{b,d-1}s_b^{d-1}$, of $\Gamma(\zeta_b^{d-1})$, we get $d-1$ linearly independent elements $B = \{s_p s_b^{d-1}, s_p f_{b,2}s_b^{d-1}, \cdots, s_p f_{b,d-1}s^{d-1}\}$ of $\Gamma(\zeta_p \zeta_b^{d-1})$. To find a basis for $\Gamma(\zeta_p \zeta_b^{d-1})$ we need to select one more element of $\Gamma(\zeta_p \zeta_b^{d-1})$ which, together with the $d-1$ elements in B, forms a linearly independent set. Let s' be any section in $\Gamma(\zeta_p \zeta_b^{d-1})$ not lying in the span of B in $\Gamma(\zeta_p \zeta_b^{d-1})$. Then there is an s which is s' minus some linear combination of the elements of B, which vanishes at b with order $\nu_b(s)$ and $\nu_b(s) = d-2$ or $\nu_b(s) > d-1$. S cannot vanish at b with order larger than d by Proposition 4.1 since then $d = c_1(L) = \Sigma \nu_p(s) \geq \nu_b(s) > d$. So $\nu_b(s) = d-2$ or $\nu_b(s) = d$. But if $\nu_b(s) = d$ then s vanishes only at b and with order d, so that by Proposition 4.3, $L = \zeta_b^d$ which contradicts the assumption $p \neq b$. Thus $\nu_b(s) = d-2$.

From knowing s and the set B, it is clear that $j_b^k : \Gamma(\zeta_p \zeta_b^{d-1}) \rightarrow J^k(\zeta_p \zeta_b^{d-1})$ is surjective for $k \leq d-1$ and injective for $k > d-1$. Thus b is not in any $W_k(L)$.

A minor modification of the above yields the same result for L with $d = 2$.

Thus we have shown

Proposition 4.6. *If* $c_1(L) = \gamma(L) = d > 1$, *then for each* b *in* M, *there is a unique* p *in* M *such that* $L = \zeta_p \zeta_b^{d-1}$, b *is not in* $W_k(L)$ *for* $k \neq d-1$, *and* b *is in* $W_{d-1}(L)$ *if and only if* $p = b$.

Example 4. Similar arguments give the following (not as nice) results for line bundles L on Riemann surfaces of genus $g = 2$.

If $c_1(L) = 1$, then $\gamma(L) = 0$ or $\gamma(L) = 1$ (Proposition 4.2). If $\gamma(L) = 1$, then $W_0(L)$ consists of one point p and $L = \zeta_p$. Also, $W_k(L) = \varnothing$ for all $k = 0$,

If $c(L) = 2$ then either L is or is not the canonical line bundle K of M.

If $L = K$, then $\gamma(K) = 2$ (Proposition 4.2). From classical results (see [G, p. 229]) we know that M is hyperelliptic, there are six distinct points w_i, $1 \leq i \leq 6$, in $W_1(K)$, and there are no points in $W_k(K)$ where $k \neq 1$. We point out also that for each p in M, there exists exactly one q in M such that $K = \zeta_p \zeta_q$, and that $p = q$ if and only if it is in $W_1(K)$.

If $c(L) = 2$ and $L \neq K$, then $\gamma(L) = 1$ (Proposition 4.2) and $L = \zeta_p \zeta_q$, where p and q are unique. Then for $p \neq q$, we have $W_0(L) = \{p, q\}$ and $W_k(L) = \varnothing$ for $k \neq 0$. If $p = q$ then $W_0(L) = W_1(L) = \{p\}$ and $W_k(L) = \varnothing$ for $k \neq 0, 1$.

If $c_1(L) = d > 2$, then $\gamma(L) = d - 1$ (Proposition 4.2). Then there are $2(d-1)^2$ points in $W_{d-2}(L)$ (counted with multiplicities as zeros of $\wedge^{d-1} j^{d-2}$). To find these points, we proceed by noting that, for any point b in M, $L \cdot \zeta_b^{2-d}$ has Chern number 2 and $L \cdot \zeta_b^{2-d} = \zeta_p \zeta_q$ where p and q are unique if and only if $\zeta_p \zeta_q$ is not the canonical bundle. By examining cases, we have the following possibilities.

Either b is not in $W_{d-2}(L)$, or, one of the following occurs:

(1) $L = \zeta_b^d$ where b is in $W_1(K)$. Then b is in $W_{d-3}(L)$ (and hence $W_{d-2}(L)$) and is not in $W_k(L)$, $k < d-3$. b is also in $W_{d-1}(L)$ but not in $W_d(L)$.

(2) $L = \zeta_w^2 \zeta_b^{d-2}$ where w is in $W_1(K)$ and b is not in $W_1(K)$. Then b is in $W_{d-3}(L)$ (and hence in $W_{d-2}(L)$) but not in $W_k(L)$, $k < d-3$. b is not in $W_{d-1}(L)$.

(3) $L = \zeta_p \zeta_b^{d-1}$ where $p \neq b$ and $\zeta_p \zeta_b \neq K$. Then b is in $W_{d-2}(L)$ and not in $W_k(L)$, $k < d-2$. b is not in $W_{d-1}(L)$.

(4) $L = \zeta_b^d$ where b is not in $W_1(K)$. Then b is in $W_{d-2}(L)$ and not in $W_k(L)$, $k < d-2$. b is in $W_{d-1}(L)$ and not in $W_d(L)$.

If there are n_i distinct points b for which L can be represented in case (i) for $1 \leq i \leq 4$, then after examining the vanishing behavior of $\wedge^{d-1} j^{d-2}$ at points of each type, we have

$$3n_1 + 2n_2 + n_3 + 2n_4 = 2(d-1)^2 > 0.$$

4.4. In the next example, we examine complex analytic manifolds of higher dimension and Weierstrass points for their canonical bundles. The canonical bundle K is but one of a number of bundles whose invariants are important to the manifold in the classification of its complex structure, some others being the tangent bundle T, the cotangent bundle T^*, their products (tensor, symmetric, exterior) and their associated k-jet bundles. The Weierstrass point behavior of these bundles (such as the codimension of the W_k) can help to classify them.

Example 5. We discuss the Weierstrass points of the canonical bundle of a hypersurface in $n + 1$ dimensional complex projective space P_{n+1}.

Proposition 4.7. *If V is nonsingular (algebraic) irreducible hypersurface of degree d in P_{n+1}, l is the hyperplane line bundle on P_{n+1}, and k_V is the canonical bundle on V, we have $\Gamma(V, k_V) \simeq \Gamma(P_{n+1}, l^{d-(n+2)})$.*

Assume now that $d > n + 2$.

The general section of $l^{d-(n+2)}$ is given by a homogeneous polynomial of degree $d - (N+2)$ in the homogeneous coordinates $z_0, \cdots, z_{n+1}$: (using multi-index notation) $s = \Sigma_{|\mu|=d-(n+2)} a_\mu z^\mu$. To get the representative of s in the local coordinate system on U^i, the set of points in P_{n+1} with homogeneous coordinate $Z_i \neq 0$, take $s_i = s/z_i^{d-(n+2)}$.

The dimension of $\Gamma(P_{n+1}, I^{d-(n+2)})$ is thus $\gamma(k_V) = C(n + m + 1, n + 1)$ $= C(d - 1, n + 1)$, where $m = d - (n + 2)$.

Then, we have the

Proposition 4.8. *For* $k \leq m$, $R_k = C(n+k, n) < \gamma(k_V) = C(n + m + 1, n + 1)$ *and* $W_k(V) = \emptyset$.

The next step then is to consider whether a p in M is in $W_{m+1}(k_V)$.

To examine this further, consider the homogeneous polynomial degree d, $P(z) = \sum_{i=0}^{n+1} z_i^d$, where $d > n + 2$. After some work, we have the following results:

If $n = 1$, and $d = 4$, then $m = 1$ and we have $W_0(k_V) = W_1(k_V) = \emptyset$, $W_2(k_V)$ $= \{\langle 1, 0, \alpha \rangle, \langle 0, 1, \alpha \rangle, \langle 1, \alpha, 0 \rangle$: where d is a fourth root of $-1\}$. $R_2 = 3 = \gamma(k_V)$.

If $n = 2$, and $d = 5$, then $m = 1$ and we have $W_0(k_V) = W_1(k_V) = \emptyset$, $W_2(k_V) = \{\langle 1, 0, 0, \alpha \rangle, \langle 1, 0, \alpha, 0 \rangle, \langle 1, \alpha, 0, 0 \rangle, \langle 0, 1, 0, \alpha \rangle, \langle 0, 1, \alpha, 0 \rangle, \langle 0, 0, 1, \alpha \rangle$: where α is a fifth root of $-1\}$. Here, $R_2 = 6 > \gamma(k_V) = 4 > R_1 = 3$.

If $n = 2$, and $d = 6$, then $m = 2$ and we have $W_0(k_V) = W_1(k_V) = W_2(k_V) = \emptyset$, and $W_3(k_V)$ is a reducible divisor. Here $R_3 = 10 = \gamma(k_V)$.

4.5. Example 6. In this example, we note that

Proposition 4.9. *The tangent bundle of projective space of n dimension has* $W_k(TP_n) = \emptyset$ *for all* k.

Furthermore, we remark that if M is a homogeneous space and E is a homogeneous vector bundle over M, then for any invariant subspace H of $\Gamma(E)$, for example $\Gamma(E)$, we have $W_k(H)$ is equal to either all of M or the empty set.

BIBLIOGRAPHY

[G] R. C. Gunning, *Lectures on Riemann surfaces*, Princeton Math. Notes, Princeton Univ. Press, Princeton, N. J., 1966. MR 34 #7789.

[K] S. Kobayashi, *Geometry of bounded domains*, Trans. Amer. Math. Soc. 92 (1959), 267–290. MR 22 #3017.

[Lw] J. Lewittes, *Differentials and metrics on Riemann surfaces*, Trans. Amer. Math. Soc. 139 (1969), 311–318. MR 38 #6050.

[Pa] R. Palais, *Seminar on the Atiyah-Singer index theorem*, Ann. of Math. Studies, no. 57, Princeton Univ. Press, Princeton, N. J., 1965. MR 33 #6649.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CALIFORNIA 92307